BOUNDARY ELEMENT METHOD APPLIED TO AXIAL AERODYNAMIC PROFILES CASCADE

Ionel Doru BACIU[1], Mircea BĂRGLĂZAN[2]

The paper presents Boundary Element Method (BEM) with linear elements for the numerical simulation of incompressible and ideal flow around reversible S axial profile cascades. Applying BEM to the Laplace equation in the stream function, the values of function, its normal derivative on the analysis domain boundary and the circulation are obtained. Next, applying BEM to the Laplace equation in the stream function and in the velocity potential function, the hydrodynamic field, the velocity field and the pressure field inside of the domain are obtained. The method was applied to a family of "S" axial turbine cascades.

Keywords: Boundary Element Method (BEM), Γ boundary, Laplace equation, linear elements, essential condition, natural condition.

1. BEM theoretic basis with linear elements

BEM is a digital algorithm for approximate solution of Laplace equation in a closed and limited plane domain $\overline{\Omega}$, with boundary conditions by Laplace equation transformation from an integral equation given on boundary domain.

We take Ω as a limited domain from R^2 Euclidian plane and Γ his boundary, which means that $\overline{\Omega} = \Omega \cup \Gamma$. We suppose that Γ boundary is some pieces smooth, which means that it has a limited number of angular points. That means that tangent at Γ boundary is continual varying, excepting in a limited number of P_1, P_2..., $P_r \in \Gamma$ points. The same behavior has the normal $\vec{n}$, on Γ boundary (fig.1).

We consider on Ω domain the Laplace equation: $\Delta u = 0$. We can demonstrate that any solution *u*, of Laplace equation, checks the integral equation:

$$c(\zeta)\, u(\zeta) = \int_\Gamma q(z)\, u^*(z,\zeta) ds_z - \int_\Gamma u(z) q^*(z,\zeta) ds_z, \quad \text{where } \zeta \in \overline{\Omega}. \tag{1}$$

In relation (1), we use the notations:

- $q(z) = \partial u(z)/\partial n$, with $z \in \Gamma$, is normal derivate of *u* function;

[1] Assistant Prof., Hydraulic Machinery Department, Faculty of Mechanical Engineering, "Politehnica" University of Timisoara, Romania
[2] Prof., Hydraulic Machinery Department, "Politehnica" University of Timisoara

- $u^*(z,\zeta) = 1/2\pi \ln 1/r(\zeta,z)$ is fundamental solution of Laplace equation attached to ζ point and defined on $R^2 \setminus \{\zeta\}$, verifying on this assemblage, the Laplace equation $\Delta u^* = 0$;
- $r(\zeta,z) = \sqrt{(x-\xi)^2 + (y-\eta)^2}$ with $\zeta = \zeta(\xi,\eta)$ and $z = z(x,y)$;
- $q^*(z,\zeta) = \partial^* u(z,\zeta)/\partial z$;
- $c(\zeta) = \begin{cases} 1 - \quad if\ \zeta \in \Omega, \\ \alpha/2\pi\ -\ if\ \zeta \in \Gamma, \text{ where } \alpha \text{ is the angle between tangent} \\ \qquad \text{on left and tangent on right, in } \zeta \text{ point;} \\ 1/2\ -\ if\ \zeta \in \Gamma \text{ and tangent on left is equal with} \\ \qquad \text{tangent on right in } \zeta \text{ point.} \end{cases}$

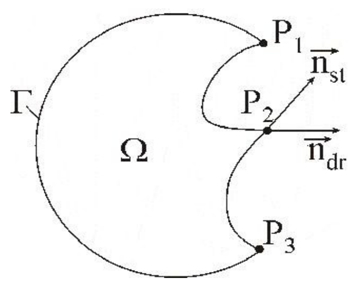

Fig. 1. P_2 angular point on Γ boundary.

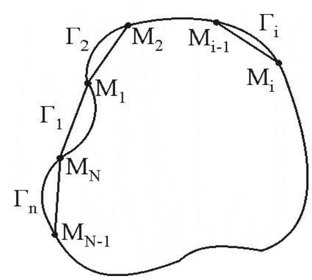

Fig. 2. Γ boundary discretization.

Laplace equation and integral equation (1) have an infinite solution; to determine a certain solution imposed condition on u function Γ boundary and $q=\partial u/\partial n$ derivate. If it impossed a_ζ value on u function in a point ζ Є Γ, $u(\zeta) = a_\zeta$ conditions, is named essential condition and $q(\zeta) = b_\zeta$ condition, is named natural condition. Generally, we have a mixt problem, like:

$$\begin{cases} u(\zeta) = f(\zeta) - \text{ pentru } \zeta \in \Gamma_1 \\ q(\zeta) = g(\zeta) - \text{ pentru } \zeta \in \Gamma_2 \end{cases} \tag{2}$$

with $\Gamma = \Gamma_1 \cup \Gamma_2$, $f(\zeta)$ and $g(\zeta)$ are functions given on Γ_1 and Γ_2 boundaries.

Having the integral equation (1) with conditions on limit (2) we can digital resolve it, meaning applying BEM It is enough to resolve equation (1) on Γ boundary because if $\zeta \in \Omega$, then $c(\zeta) = 1$, and relation (1) becomes:

$$u(\zeta) = \int_\Gamma q(z)\, u^*(z,\zeta)\, ds_z - \int_\Gamma u(z)\, q^*(z,\zeta)\, ds_z, \text{ where } \zeta \in \Omega. \tag{3}$$

The relation (3), permits us to calculate u function in every point of Ω domain, because the elements under the integral are known after we solve equation (1) on Γ.

BEM application with linear elements imposes two types of simultaneous approximations:

1. Boundary meshing (discretization) – that means to take N points $M_1, M_2, ..., M_N$ on Γ boundary; we noted with Γ_i for $\overline{M_{i-1}, M_i}$ segments; so, we replace Γ boundary with $\overline{\Gamma} = \bigcup_{i=1}^{N} \Gamma_i$ - a polygonal line (fig. 2);
2. The linear approximation of u and $q = \partial u / \partial n$ functions on every Γ_i element; Γ_i $\left(i = \overline{1, N} \right)$ segments are named boundary elements.

We note x_i and y_i the coordinates of M_i points. If on Γ_i boundary element we take $t \in [-1,1]$, so for M_{i-1} point, t=-1 and for M_i point, t=1, the coordinates of a point $M_j(x,y) \in \Gamma_j$, is verifying the relation:

$$\begin{cases} x = \frac{1}{2}\left(x_j - x_{j-1}\right) t + \frac{1}{2}\left(x_j + x_{j-1}\right) \\ y = \frac{1}{2}\left(y_j - y_{j-1}\right) t + \frac{1}{2}\left(y_j + y_{j-1}\right) \end{cases} \quad \text{or}$$

$$\begin{cases} x = \frac{1}{2}\ x_{j-1}\left(1 - t\right) + \frac{1}{2}\ x_j\left(1 + t\right) \\ y = \frac{1}{2}\ y_{j-1}\left(1 - t\right) + \frac{1}{2}\ y_j\left(1 + t\right) \end{cases} \tag{4}$$

We take u_i and q_i unknown values of u and q functions in points M_i ($i = \overline{1 - N}$). We suppose that u and q have a linear variation function of t on Γ_j boundary for every $t \in [-1,1]$. For u and q, we have the relations:

$$\begin{cases} u(t) = u_{j-1}\ \varphi_1(t) + u_j\ \varphi_2(t) \\ q(t) = q_{j-1}\ \varphi_1(t) + q_j\ \varphi_2\ (t) \end{cases} \tag{5}$$

We replace in equation (1), Γ boundary with $\overline{\Gamma} = \bigcup_{i=1}^{N} \Gamma_i$ and u and q functions with (5) expression. It results:

$$c(\zeta)\, u(\zeta) = \sum_{j=1}^{N} \left[q_{j-1} \int_{\Gamma_j} \varphi_1\, u^*\, ds + q_j \int_{\Gamma_j} \varphi_2\, u^*\, ds + \right] -$$

$$- \sum_{j=1}^{N}\left[u_{j-1}\int_{\Gamma_j}\varphi_1 q^* ds + u_j\int_{\Gamma_j}\varphi_2 q^* ds + \right] =$$

$$= \sum_{j=1}^{N} q_j\left[\int_{\Gamma_{j+1}}\varphi_1 u^* ds + \int_{\Gamma_j}\varphi_2 u^* ds\right] - \sum_{j=1}^{N} u_j\left[\int_{\Gamma_{j+1}}\varphi_1 q^* ds + \int_{\Gamma_j}\varphi_2 q^* ds\right] \quad (6)$$

where $\Gamma_{N+1} = \Gamma_1$ and u^* and q^* depends on $\zeta \in \overline{\Omega}$ variable.

Now, we replace in relation (6), ζ one by one, with the values of M_1, M_2, ..., M_N points and we note:

$$\begin{cases} c_i = c_i(M_i), \quad \text{where } i = \overline{1,N} \\ H_{ij} = \delta_{ij} c_i + \left[\int_{\Gamma_{j+1}}\varphi_1 q^*(M_i) ds + \int_{\Gamma_j}\varphi_2 q^*(M_i) ds\right]. \\ G_{ij} = \int_{\Gamma_{j+1}}\varphi_1 u^*(M_i)\, ds + q_j\int_{\Gamma_j}\varphi_2 u^*(M_i) ds \end{cases} \quad (7)$$

So, (1) equation becomes a linear system of equations:

$$\sum_{j=1}^{N} G_{ij}\, q_j - \sum_{j=1}^{N} H_{ij}\, u_j = 0 \qquad \text{for } i = \overline{1,N} \quad (8)$$

In this system, the unknown are: $u_1,...,u_N$ and their derivates $q_1,...,q_N$, G_{ij} and coefficients H_{ij}, are some curvilinear integrals (on Γ_j segments), from known functions; they become some definite integrals between -1 and 1 reporting to t parameter.

If $i \neq j$ the integrals which give the values of G_{ij} and H_{ij} do not have singularities points and they can be calculated (by Gauss method – for example).

If $i=j$, the integrals are improper and they can be calculated so:

- G_{ii} - there are directly calculated, and it obtain:

$$G_{ii} = \frac{l_i}{2}\left[\frac{3}{2} - \ln(l_i)\right] + \frac{l_{i+1}}{2}\left[\frac{3}{2} - \ln(l_{i+1})\right], \quad i = \overline{1,N} \quad (9)$$

where l_i is length of segment Γ_i ($l_{N+1} = l_1$);

- H_{ij} there are indirectly calculated, knowing that $u(z) \equiv 1$ function with $z \in \overline{\Omega}$, is verifying $\Delta u = 0$ Laplace equation; so, from relation (8) we obtain relation (10):

$$\sum_{j=1}^{N} H_{ij} = 0, \quad i = \overline{1,N} \qquad \text{or} \qquad H_{ii} = -\sum_{i=1,\, j\neq i}^{N} H_{ij} \quad (10)$$

So, (1) integral equation on Γ boundary, is reduced to a N system equations in $2N$ unknowns, the approximate values of unknown u and q functions, in M_1, M_2, ...M_N points.

The solution of Laplace equation depends on N parameters. To determine a given solution for a concrete problem in relation (8) it is generally imposed the value of K sizes u_i and N-K sizes q_j. Generally if to relation (8) we add N linear relation (limit mixt condition – essentials and naturals) by form:

$$\sum_{j=1}^{N}\left(\alpha_{ij}\,u_j + \beta_{ij}\,q_j\right) = \gamma_i, \quad i = \overline{1,N}\,. \tag{11}$$

then relation (8) have unique solution.

2. Calculation of *u* function and its derivates *q* inside Γ boundary

We take $M_0\left(\zeta_0 = (x_0,\ y_0)\right) \in \Omega$. In this case $c(\zeta_0) = 1$. The relation (6) for $\zeta = \zeta_0$, becomes:

$$u(\zeta_0) = \sum_{j=1}^{N} G_{0j}\,q_j - \sum_{j=1}^{N} H_{0j}\,u_j \tag{12}$$

where:

$$\begin{cases} G_{0j} = \int_{\Gamma_{j+1}} \varphi_1\,u^*(\zeta_0)\,ds + q_j \int_{\Gamma_j} \varphi_2\,u^*(\zeta_0)ds \\ H_{0j} = \int_{\Gamma_{j+1}} \varphi_1\,q^*(\zeta_0)ds + \int_{\Gamma_j} \varphi_2\,q^*(\zeta_0)ds \end{cases} \tag{13}$$

The integrals form relations (13) are deduced to non-singular definite integrals $\left(\text{for } t \in [-1, 1]\right)$ and are calculated identically like H_{ij} and G_{ij} coefficients $(\text{with } i \neq j)$. Because u^* and q^* are indefinite derivable in every $\zeta_0 \in \Omega$, in integrals from equations (13) it can be derivates under the integral every time we want. From (12) and relations (13) results:

$$\frac{\partial^{m+n} u(\zeta_0)}{\partial x_0^m\,\partial y_0^m} = \sum_{j=1}^{N} G_{0j}^{(m,n)}\,q_j - \sum_{j=1}^{N} H_{0j}^{(m,n)}\,u_j \tag{14}$$

where:

$$\begin{cases} G_{0j} = \int_{\Gamma_{j+1}} \varphi_1 \dfrac{\partial^{m+n} u^*(\zeta_0)}{\partial x_0^m\,\partial y_0^m}\,ds + q_j \int_{\Gamma_j} \varphi_2 \dfrac{\partial^{m+n} u^*(\zeta_0)}{\partial x_0^m\,\partial y_0^m}\,ds \\ H_{0j}^{(m,n)} = \int_{\Gamma_{j+1}} \varphi_1 \dfrac{\partial^{m+n} q^*(\zeta_0)}{\partial x_0^m\,\partial y_0^m}\,ds + \int_{\Gamma_j} \varphi_2 \dfrac{\partial^{m+n} q^*(\zeta_0)}{\partial x_0^m\,\partial y_0^m}\,ds \end{cases} . \tag{15}$$

Because the analytical expression of partial derivatives $\partial^{m+n}u^*(\zeta_0)/\left(\partial x_0^m\,\partial y_0^m\right)$, $\partial^{m+n}q^*(\zeta_0)/\left(\partial x_0^m\,\partial y_0^m\right)$ is complex, we prefer to calculate u function by digital derivations. This, because we can calculate the values of u functions in points as close as can be, using relations (13).

3. Numerical results

The method was applied to a reversible axial "S" turbine cascade. In figures 3–16, we present the results for an axial reversible "S" turbine cascade (with modulation of chord at a_p=0.7l) composed of profiles from the NACA 4412 class with the parameters: t/l = 0.8322; a=b = 0.4161; β^{AM} = 48.48°; β_s = 32.99°. After computations, we obtained: β^{AV}= 34.71° and Γ^t = 0.37. All parameters are non-dimensional.

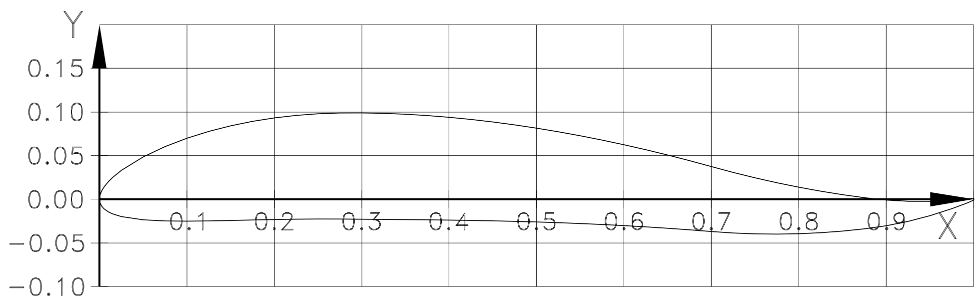

Fig. 3. The reversible profile NACA in "S".

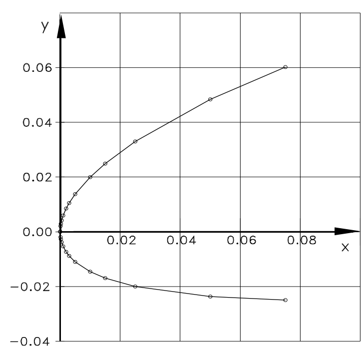

Fig. 4. Leading edge NACA S profile.

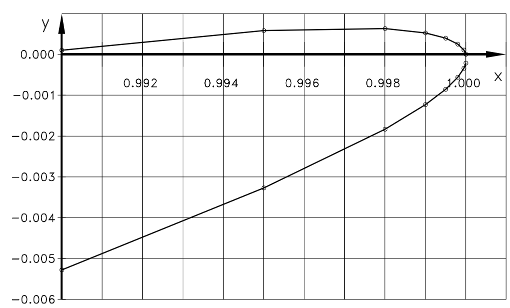

Fig. 5. Trailling edge zone NACA S profile.

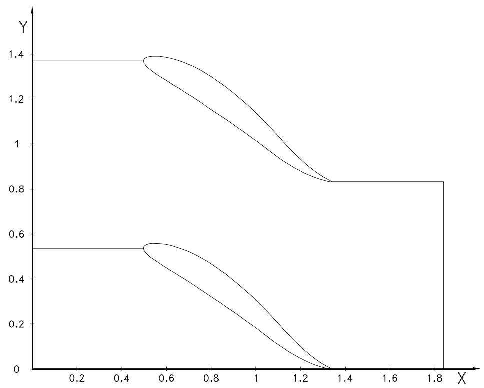

Fig. 6. The domain of analysis.

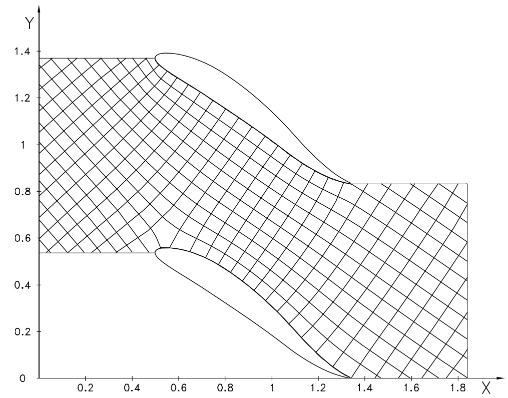

Fig. 7. The hydrodynamic field.

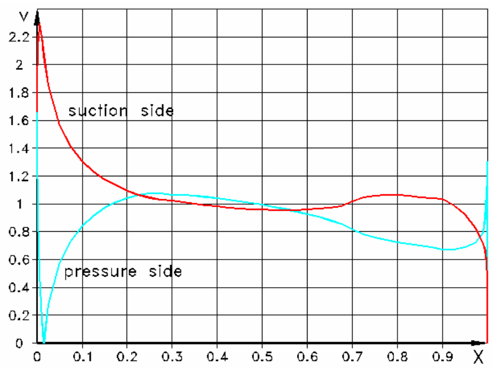

Fig. 8. The velocity field on the profile boundary.

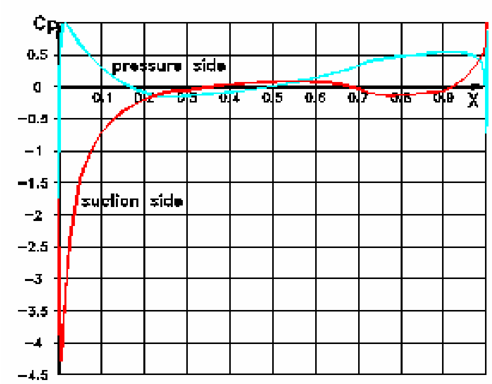

Fig. 9. The pressure field on the profile boundary.

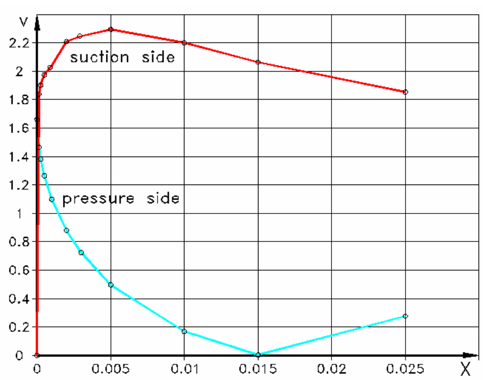

Fig. 10. The velocity field on the profile boundary in the leading edge zone.

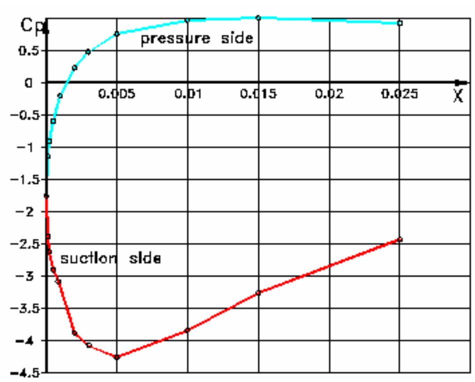

Fig. 11. The pressure field on the profile boundary in the leading edge zone.

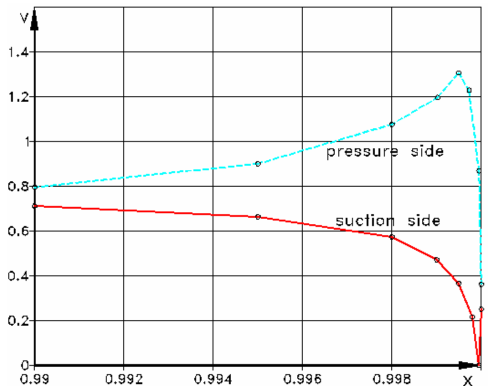

Fig. 13. The velocity field on the profile boundary in the trailling edge zone.

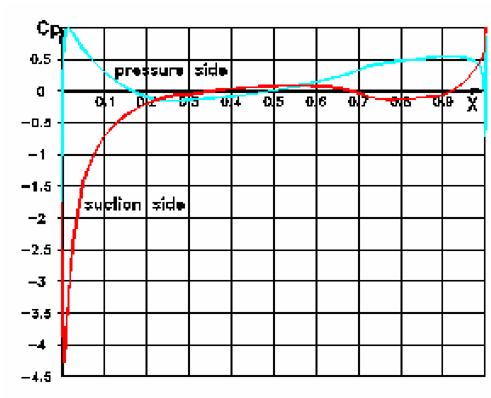

Fig. 14. The pressure field on the profile boundary in the trailling edge zone

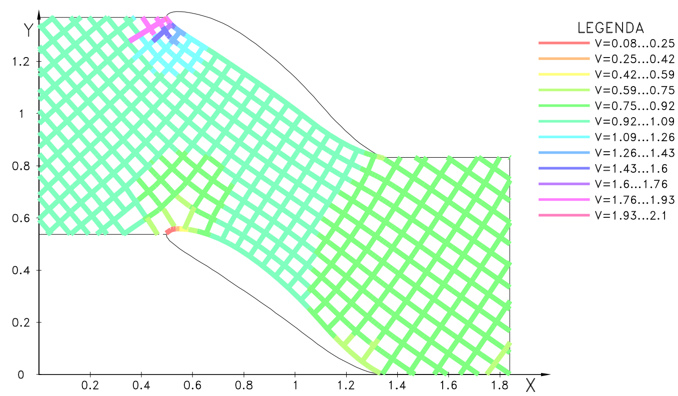

Fig. 15. The speed field in the domain of analysis.

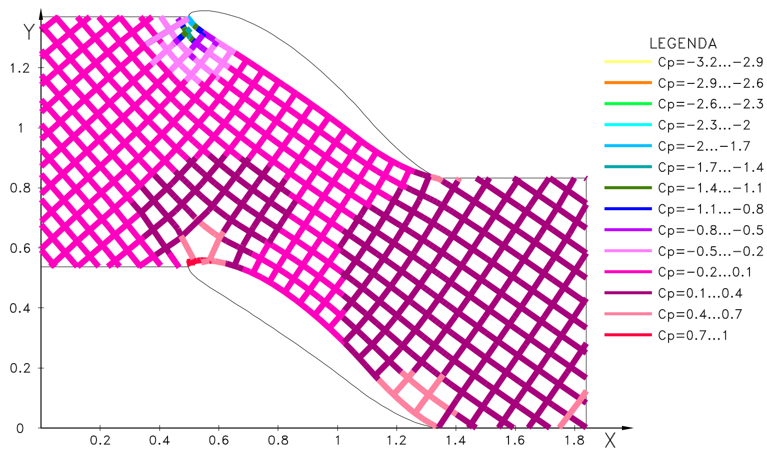

Fig. 16. The pressure field in the domain of analysis.

4. Conclusions

Using of BEM method, with linear elements, in hydrodynamic turbo-machines proved to be efficient, operative and of great accuracy. It opens great perspectives and will even determine a transformation in our research and design view on these machines.

Applying BEM with linear elements to the Laplace equation in the stream function values, the values of function, its normal derivative on the analysis domain boundary and the circulation are obtained.

Finally, the hydrodynamic field, the velocity field and the pressure field inside of the domain are determined.

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