ON THE KINEMATIC OF THE TRAVELER WAVE

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Starting from Gerstner's potential of traveler wave and determining its constant as function of the wave height and length, as well as of the channel water depth, one calculates variations in time of the pressure of the bi-dimensional and unsteady motion of traveler wave for any point of the fixed trihedron.

The obtained results have a great importance to establish the boundary conditions in the numerical solving of the shape-changing problem of the traveler wave in its propagation on an inclined bottom.

Keywords: Traveller wave. Wave kinematics. Pressure distribution in wave.

1. Gerstner's wave potential as direct method in Hydrodynamics

In the present difficult struggle [1-6] against the beaches erosion on the Romanian seaside, the traveller wave potential, obtained by Gerstner in 1802 [7] has not only a

$$\Phi(X,Y,t) = F(Y)\cdot\sin(kX-\omega t), \tag{1}$$

theoretical, but also a practical importance for the study of sand particle motion.

To verify the mass conservation equation, which gives to this velocity potential the quality to by an harmonic function $U'_{\mathrm{X}} + V'_{\mathrm{Y}} = \Phi''_{\mathrm{X}^2} + \Phi''_{\mathrm{Y}^2} = 0$, Gerstner determined from the characteristic equation $r^2 - k^2 = 0 \rightarrow r = \pm\, k$, associated to the Euler differential equation with constant coefficients $F''_{\mathrm{Y}^2}(Y) - k^2 F(Y) = 0$, the general expression of the wave amplitude function $F(Y) = Ae^{\mathrm{kY}} + Be^{\mathrm{-kY}}$, in which he could determine one of the constant considering that on the channel bottom the vertical component V of the velocity must be zero

$Y \quad \eta = Y\text{-}H$

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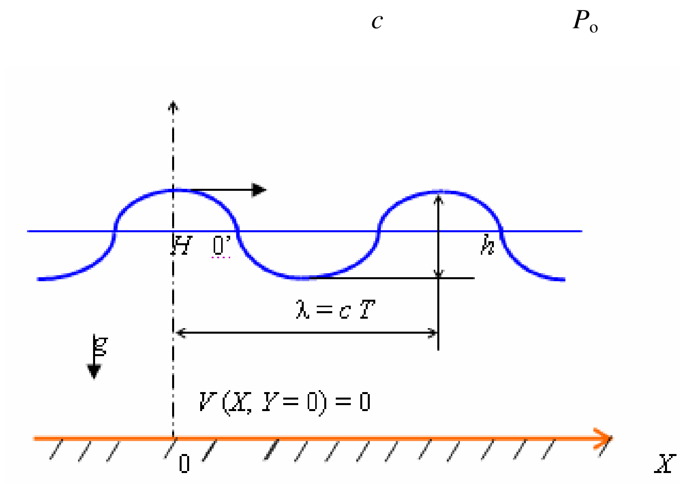

Fig. 1. The coordinate axis and the wave kinematics' parameters.

$$V\big|_{Y=0} = \Phi'_Y\big|_{Y=0} = k\left(A-B\right)\sin\left(kX-\omega t\right) = -k\,C\,\mathrm{sh}k\left(Y\right)\sin\left(kX-\omega t\right) = 0\,, \quad (2)$$

with $A = B$, the Gerstner's traveller wave potential taking the form

$$\Phi\left(X,Y,t\right) = C\,\mathrm{ch}\,k\left(Y\right)\sin\left(kX-\omega t\right). \quad (3)$$

2. Bernoulli's relation for the unsteady motion of ideal and heavy fluid

From the two unsteady and non-rotational ($U'_Y - V'_X = 0$) motion equations of an ideal and heavy liquid

$$0 = U'_t + UU'_X + VU'_Y + \frac{1}{\rho}P'_X \pm \left\{VV'_X\right\}, \quad (4)$$

$$-g = V'_t + UV'_X + VV'_Y + \frac{1}{\rho}P'_Y \pm \left\{UU'_Y\right\}, \quad (5)$$

we obtain by differential addition and then their integration, with $\eta = Y\text{-}H$, the relation

$$\frac{1}{g}\Phi'_t + \frac{U^2+V^2}{2g} + \frac{P}{\gamma} + \eta + H = K(t). \quad (6)$$

3. Determination of the second constant in Gerstner's wave potential

In this purpose [8] we used the Bernoulli's relation (6), in which to profit by the same constant value $K(t)$ in the whole domain, occupied by the ideal fluid, we shall use it in the same moment of time, for example, initial time $t = 0$.
Also, we considered the constant pressure P_0 on the wave free surface and how $\eta(X, t) = Y\text{-}H$, the wave height will have consequently the expression

$$h = \eta\left(0,0\right) - \eta\left(\frac{\lambda}{2},0\right) = Y\left(0,0\right) - Y\left(\frac{\lambda}{2},0\right) = H + \frac{h}{2} - \left(H - \frac{h}{2}\right) = \quad (7)$$

$$= \frac{U^2\left(\frac{\lambda}{2}, H-\frac{h}{2}, 0\right) - U^2\left(0, H+\frac{h}{2}, 0\right)}{2g} + \frac{1}{g}\left[\Phi'_t\left(\frac{\lambda}{2}, H-\frac{h}{2}, 0\right) - \Phi'_t\left(0, H+\frac{h}{2}, 0\right)\right],$$

where we have took into consideration the relation $V(0,H+h/2,0) = V(\lambda/2,H\text{-}h/2,0) = 0$ for the vertical velocity deduced from (3), and where by calculation of the maximum and minimum values of the horizontal velocity at the wave free surface

$$U\left(0, H+\frac{h}{2}, 0\right) \cong kC\,\mathrm{ch}k\left(H+\frac{h}{2}\right) \text{ and } U\left(\frac{\lambda}{2}, H-\frac{h}{2}, 0\right) \cong -kC\,\mathrm{ch}k\left(H-\frac{h}{2}\right), \quad (8)$$

and by introduction of the velocity potential derivatives with respect to the time

$$\Phi_t'\left(0, H+\frac{h}{2}, 0\right) \cong -\omega C\,\mathrm{ch}k\left(H+\frac{h}{2}\right) \text{ and } \Phi_t'\left(\frac{\lambda}{2}, H-\frac{h}{2}, 0\right) \cong \omega C\,\mathrm{ch}k\left(H-\frac{h}{2}\right), \quad (9)$$

the relation (7) will lead as to an equation of 2nd degree, from that we can determinate the expression of the constant C (h, λ, H)

$$k^2\left[\mathrm{ch}^2k\left(H+\frac{h}{2}\right)-\mathrm{ch}^2k\left(H-\frac{h}{2}\right)\right]C^2-2\omega\left[\mathrm{ch}k\left(H+\frac{h}{2}\right)+\mathrm{ch}k\left(H-\frac{h}{2}\right)\right]C+2gh=0,$$

its solution being in the general case

$$C(\lambda,H,h)=\frac{\omega\left[\begin{array}{l}\mathrm{ch}k\left(H+h/2\right)+\\+\mathrm{ch}k\left(H-h/2\right)\end{array}\right]\pm\sqrt{\begin{array}{l}\omega^2\left[\mathrm{ch}k\left(H+h/2\right)+\mathrm{ch}k\left(H-h/2\right)\right]^2-\\-2ghk^2\left[\mathrm{ch}^2k\left(H+h/2\right)-\mathrm{ch}^2k\left(H-h/2\right)\right]\end{array}}}{k^2\left[\mathrm{ch}^2k\left(H+h/2\right)-\mathrm{ch}^2k\left(H-h/2\right)\right]}, \quad (10)$$

and consequently, the motion potential (3) will have the implicit expression

$$\Phi(X,Y,t,\lambda,H,h)=C(\lambda,H,h)\,\mathrm{ch}\,kY\sin(kX-\omega t). \quad (3')$$

With a view to obtain non-imaginary solutions for the constant C, it will must that in the relation (10) the expression value under the root must be positive. In this case, denoting by $\aleph = H/\lambda$ and $\chi = h/\lambda$ the relative heights of the wave and of the static water level in the channel, after a simple calculus we obtain the condition

$$\mathrm{ch}2\pi\left(\aleph+\frac{\chi}{2}\right)+\mathrm{ch}2\pi\left(\aleph-\frac{\chi}{2}\right)\geq\frac{2gh}{c^2}\left[\mathrm{ch}2\pi\left(\aleph+\frac{\chi}{2}\right)-\mathrm{ch}2\pi\left(\aleph-\frac{\chi}{2}\right)\right], \quad (11)$$

or taking into account of the over-unitary value of the ratio calculated from (17) and of its expression in infinite series by performing the division, we obtained the evaluation

$$1\prec\frac{\mathrm{ch}2\pi\left(\aleph+\chi/2\right)}{\mathrm{ch}2\pi\left(\aleph-\chi/2\right)}\leq\frac{\frac{2gh}{c^2}+1}{\frac{2gh}{c^2}-1}=\frac{x+1}{x-1}\cong 1+2\left(\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\frac{1}{x^4}\cdots\right)\succ 1, \quad (12)$$

in which we denoted in the second ratio, over-unitary in practice, by $x = 2gh/c^2 \succ 1$, where by introducing of the propagation velocity expression of the traveller wave, obtained from the Gerstner's theory, $c^2=\frac{g\lambda}{2\pi}\,\mathrm{th}\,2\pi\aleph$, results the necessary condition to fulfil between the relative values, reported to the wave length, of the wave height and the water depth in the channel (fig.2), relation that stipulate the obtaining of the real values for the constant $C > 0$, because ch k $(H + h/2) >$ ch k $(H - h/2)$

$$\chi_{\max} = 0.08 \geq \chi \succ \frac{1}{4\pi} \text{th } 2\pi \aleph . \tag{13}$$

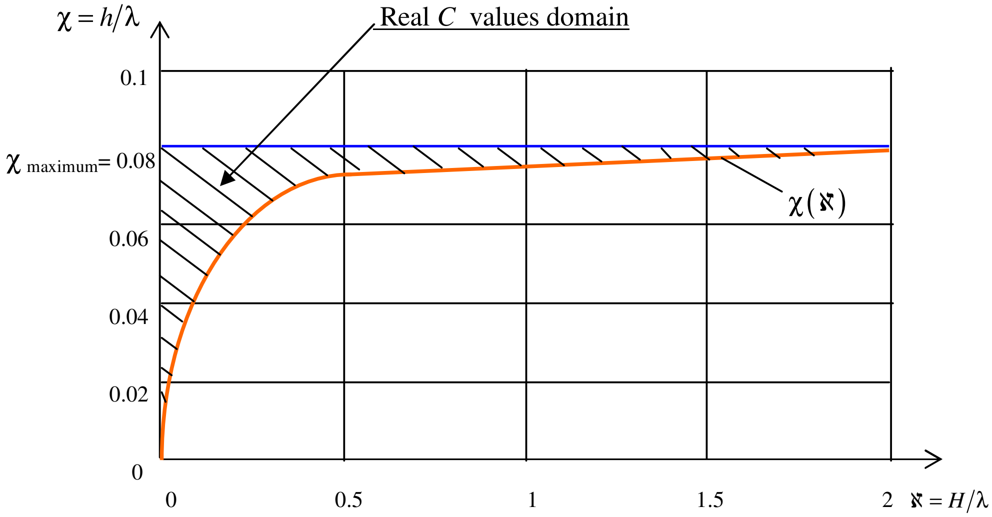

Fig. 2 The dependence of the relative wave height $\chi = h/\lambda$ of the water relative depth in the channel $\aleph = H/\lambda$

4. Pressure local variation at wave propagation on a right bottom

To determinate the pressure variation at wave propagation, we have two possibilities:

4.1 Starting from the unsteady motion equations of the ideal and heavy liquid (4) and (5), by replacing the velocity expressions and their differential calculated from the potential (3) the two motion equations becoming:

$$P'_X = -\rho\omega kC \,\text{ch}kY \sin(kX - \omega T) + \rho k^3 C^2 \sin(kX - \omega T)\cos(kX - \omega T), \tag{4'}$$

$$P'_Y = -\gamma + \rho\omega kC \,\text{sh}kY \cos(kX - \omega T) - \rho k^3 C^2 \text{sh}kY \,\text{ch}kY, \tag{5'}$$

which by partial integration give us two relations for the same pressure function

$$P(X,Y,T) = F(Y,T) + \rho\omega C \,\text{ch}kY \cos(kX - \omega T) - \frac{\rho k^2 C^2}{4} \cos 2(kX - \omega T), \tag{4''}$$

$$P(X,Y,T) = G(X,T) - \gamma Y + \rho\omega C \,\text{ch}kY \cos(kX - \omega T) - \frac{\rho k^2 C^2}{4} \text{ch}2kY . \tag{5''}$$

Because the pressure condition $P_{\mathrm{FS}}\left[\overset{\mathrm{o}}{Y}\left(X,T\right)\right]=P_0=\text{constant}$, on the free surface is not proper to the two unknown functions F and G, taking into consideration of their variables, we shall calculate the partial differential of the pressure function from the relations (5") and (4"):

$$P'_{\mathrm{X}}=G'_{\mathrm{X}}-\rho k\omega C\,\mathrm{ch}\,kY\sin\left(kX-\omega T\right),\ P'_{\mathrm{Y}}=F'_{\mathrm{Y}}+\rho k\omega C\,\mathrm{sh}\,kY\cos\left(kX-\omega T\right), \quad (14)$$

which by matching with their expressions from (4') and (5') offer us the partial differentials of the two unknown functions from the previous partial integration, under the form:

$$G'_{\mathrm{X}}=\rho k^3C^2\sin\left(kX-\omega T\right)\cdot\cos\left(kX-\omega T\right)=\frac{\rho k^3C^2}{2}\sin 2\left(kX-\omega T\right), \quad (15)$$

$$F'_{\mathrm{Y}}=-\gamma-\rho k^3C^2\,\mathrm{sh}\,kY\,\mathrm{ch}\,kY=-\gamma-\frac{\rho k^3C^2}{2}\mathrm{sh}\,2kY, \quad (16)$$

from where, by partial integration, we shall obtain their expression with the approximation of two constants:

$$G\left(X,T\right)=G_0-\frac{\rho k^2C^2}{4}\cos 2\left(kX-\omega T\right) \text{ and } F\left(Y\right)=F_0-\gamma Y-\frac{\rho k^2C^2}{4}\mathrm{ch}\,2kY. \quad (17)$$

Replacing these so determined two functions in the relations (5") and (4"), we shall obtain the two identically expressions of the unique function of pressure, considering that $F_0 = G_0$:

$$P\left(X,Y,T\right)=F_0-\gamma Y+\rho\omega C\,\mathrm{ch}kY\cos\left(kX-\omega T\right)-\frac{\rho k^2C^2}{4}\left[\mathrm{ch}2kY+\cos 2\left(kX-\omega T\right)\right]$$

and (18)

$$P\left(X,Y,T\right)=G_0-\gamma Y+\rho\omega C\,\mathrm{ch}kY\cos\left(kX-\omega T\right)-\frac{\rho k^2C^2}{4}\left[\mathrm{ch}2kY+\cos 2\left(kX-\omega T\right)\right].$$

To determination of this unique constant, we shall observe that the cancelling of the component $V = 0$, can have place, so much on the channel bottom $Y = 0$, how on the wave free surface, if the down condition is fulfil

$$\sin\left(kX-\omega T\right)=0 \quad\rightarrow\quad kX-\omega T=0 \quad\rightarrow\quad X=\frac{\lambda}{T_0}T, \quad (19)$$

which is not convenient, because the time can not more flow, when we fixed the abscise of de pressure variations observation in the phenomenon of traveller wave propagation. In this condition, we shall cancel the wave velocity component $U = 0$, which can have place if only

$$kX-\omega T=2\pi\left(\frac{X}{\lambda}-\frac{T}{T_0}\right)=\frac{\pi}{2} \quad\rightarrow\quad X=\frac{\lambda}{4}+\frac{\lambda}{T_0}T=\frac{\lambda}{4}+cT. \quad (20)$$

Putting this condition on the wave free surface, we obtain for the initial time

$$P\left(\frac{\lambda}{4},H,0\right)=P_0=F_0-\gamma H+\rho\omega C\,\mathrm{ch}kH\,\underbrace{\cos\pi/2}_{0}-\frac{\rho k^2C^2}{4}\left[\mathrm{ch}2kH+\underbrace{\cos\pi}_{-1}\right], \quad (20)$$

from where we can deduce the constant value

$$F_0=P_0+\gamma H+\frac{\rho k^2C^2}{4}\left(\mathrm{ch}2kH-1\right), \quad (21)$$

determining in this kind the function expression that give the pressure variation in the traveller wave interior by the identification of the arbitrary integration function

$$\begin{aligned}P\left(X,Y,T\right)&=P+\gamma\left(H-Y\right)+\rho\omega C\,\mathrm{ch}kY\cos\left(kX-\omega T\right)+\\&+\frac{\rho k^2C^2}{4}\left[\mathrm{ch}2kH-1-\mathrm{ch}2kY-\cos2\left(kX-\omega T\right)\right].\end{aligned} \quad (22)$$

4.2. Starting from the Daniel Bernoulli's equation, deduced from the two motion equation (6), in which by introducing the expressions of the potential time differential and spacey partial differential and velocity components, we obtain for the integration function the expression

$$\begin{aligned}K(T)&=-\omega C\,\mathrm{ch}\,kY\cos(kX-\omega T)+\frac{P}{\rho}+\gamma Y+\\&+\frac{k^2C^2}{2}\left[\mathrm{ch}^2kY\cos^2(kX-\omega T)+\mathrm{sh}^2kY\sin^2(kX-\omega T)\right]\end{aligned} \quad (23)$$

for whose determination we shall utilize the same condition for the two limits, valuable on the Free Surface for $X = 0$ and $Y_{FS} = H$, knowing the pressure value P_0 only on the wave free surface

$$K\left(\frac{\lambda}{4},\text{respectively}\,\frac{3\lambda}{4}\right)\Bigg|_{Y=H}=\frac{k^2C^2}{2}\mathrm{sh}^2kH+\frac{P_0}{\rho}+\gamma H\ , \quad (24)$$

which introduced in the Bernoulli relation (6), permit us to known the pressure variation function in the wave interior

$$\begin{aligned}P(X,Y,T)&=P_0+\gamma(H-Y)+\frac{\rho k^2C^2}{2}\mathrm{sh}^2kH+\frac{2\pi\rho C}{\lambda}\mathrm{ch}\,kY\cos(kX-\omega T)-\\&-\frac{\rho k^2C^2}{2}\left[\mathrm{ch}^2kY\cos^2(kX-\omega T)+\mathrm{sh}^2kY\sin^2(kX-\omega T)\right]\end{aligned} \quad (25)$$

that for the abscise $X = 0$ become

$$\begin{aligned}P(0,Y,T)&=P_0+\gamma(H-Y)+\frac{\rho k^2C^2}{2}\mathrm{sh}^2kH-\frac{2\pi\rho C}{\lambda}\mathrm{ch}\,kY\cos\omega T-\\&-\frac{\rho k^2C^2}{2}\left(\mathrm{ch}^2kY\cos^2\omega T+\mathrm{sh}^2kY\sin^2\omega T\right)\end{aligned} \quad (25')$$

and for the bottom $Y = 0$, having the expression

$$P(0,0,T) = P_0 + \gamma H + \frac{\rho k^2 C^2}{2}\,\mathrm{sh}^2 kH - \frac{2\pi\rho C}{\Im}\cos\omega T - \frac{\rho k^2 C^2}{2}\cos^2\omega T, \qquad (25'')$$

or in dimensionless expression, for $p = P / P_0$ and $t = T /$Á, become (fig.3)

$$p(0,0,t) = 1 + \frac{\gamma H}{P_0} + \frac{2\pi^2\rho C^2}{P_0\lambda^2}\,\mathrm{sh}^2 2\pi\frac{H}{\lambda} - \frac{2\pi\rho C}{P_0\Im}\cos 2\pi t - \frac{2\pi^2\rho C^2}{P_0\lambda^2}\cos^2 2\pi t\,. \quad (25''')$$

In this kind we can calculate the pressure variation in time at different heights and wanted abscise (for instance at $Y = Y_i$ and $X = 0$) in the period of wave propagation with a special geometry at the free surface (wavelength λ and height h), the static water depth in the channel being H.

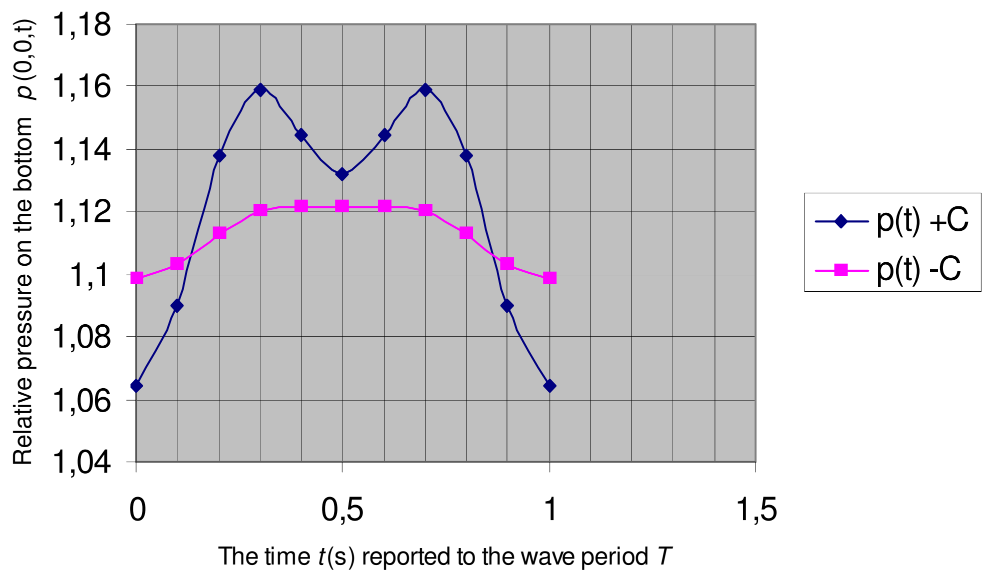

Fig. 3. Dimensionless pressure variation in a point on the channel bottom, in a period of traveller wave propagation

6. Conclusions

The manner, used to determine the pressure variations in the traveller wave propagation, prove to be an appropriate method and show at the same time, that one cannot determine the wave height shape with an apparatus, which measures the static pressure.

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